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BIFURCATION HIERARCHY OF A RECTANGULAR PLATE

K. IKEDA and M. NAKAZAWA
Department of Civil Engineering, Tohoku University, Aoba, Sendai 980-77, Japan

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Abstract—The mechanism of the complex recursive bifurcation behavior of a four-sides simply-supported rectangular plate is investigated. Such complex behavior is due to the "hidden" symmetry of the plate associated with the periodic nature of the solutions. The group-theoretic bifurcation theory is employed to arrive at a lattice of subgroups which expresses the rule for the behavior. This rule is shown to suffer degeneration due to the restriction by the boundaries compared with that of geometrical symmetry. The governing equation of this plate is discretized by means of Galerkin's method with the use of the double Fourier series as shape functions. The tangential stiffness matrix of the plate is shown to be block-diagonalized by appropriately permuting the order of the Fourier series following the rule presented. The bifurcation analysis of the plate is carried out to assess the validity of the rule and to demonstrate the merit of the block-diagonalization. As a result of this, the vital role of the bifurcation rule in the proper understanding and successful analysis of the complex bifurcation behavior has been demonstrated. © 1997 Elsevier Science Ltd

1. INTRODUCTION

Most bifurcation behavior takes place as a consequence of the (partial) loss of the symmetry of a system. The symmetry of the governing equation is described by the group-equivariance condition, which shows the objectivity of this equation. By virtue of a recent development of the group-theoretic bifurcation theory in nonlinear mathematics, the mechanism of the (symmetry-breaking) bifurcation behavior of a system equivariant to a group can be known *a priori* [e.g. Sattinger (1979, 1983); Fujii and Yamaguti (1980); Golubitsky and Schaeffer (1985); Golubitsky *et al.* (1988)]. Such bifurcation can be characterized by a bifurcation rule expressed in terms of a lattice of subgroups:

$$G \to G_1 \to G_2 \to \cdots,$$

where $G_i \to G_{i+1}$ represents the emergence of a G_{i+1}-invariant solution from a G_i-invariant one. This equation means that the symmetry of the system is reduced from a G-invariant state into G_1-, G_2-, ..., invariant ones. The framework of the bifurcation of the system can be known *a priori* by investigating the rule of bifurcation of systems equivariant to groups G, G_1, G_2, ... in a recursive manner.

In the field of fluid dynamics, the bifurcation behavioral characteristics of the Couette–Taylor flow (Taylor, 1923), the Benard convection (Koschmieder, 1966), and so on, have been untangled fully and various kinds of patterned flows are systematically classified (Schaeffer, 1980; Iooss, 1986; Crawford and Knobloch, 1991; Chossat and Iooss, 1994). In the field of mechanics, the mechanism of the formation of an echelon mode in uniform materials has been explained as the bifurcation of an $O(2) \times O(2)$-equivariant system in Ikeda *et al.* (1994).

The symmetry of systems can be categorized into "geometrical" symmetry and "hidden" one. The circumferential symmetry of axisymmetric shells serves as a typical example of the former. The latter is due to the periodic nature of solutions, and its examples can be found for in an ecological interacting and diffusing system by Fujii *et al.* (1982), in an axially-loaded cylinder by Hunt (1986), and in a beam by Goto *et al.* (1994). While the bifurcation structure of the former is determined solely from the symmetry group of the system under consideration, that of the latter is influenced by the boundary conditions.

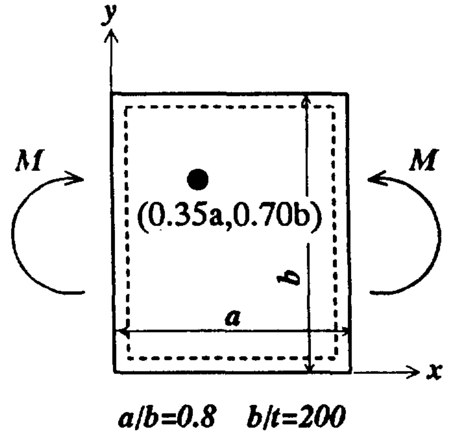

Fig. 1. Simply-supported rectangular plate under pure bending.

In the field of structural engineering, the analysis of recursive (secondary) bifurcation has been an active topic of research [e.g. Chilver (1967); Nakamura and Uetani (1979); Thompson and Hunt (1984); Maaskant and Roorda (1992); Nakazawa *et al.* (1991, 1993, 1996)]. The block-diagonalization method, by which the governing equation is decomposed into a series of independent equations, has been established as a means to exploit symmetry [e.g. Healey (1988); Zloković (1989); Dinkevich (1991); Murota and Ikeda (1991)]. Extensive research has been conducted on axisymmetric systems (equivariant to a group $C_{\infty v}$ for circular symmetry or to a group C_{nv} for regular $n$gonal symmetry) to arrive at the lattice of subgroups, and to implement the block-diagonalization method into the framework of the finite element method (Ikeda *et al.*, 1991, 1992; Ikeda and Murota, 1991). It is advantageous in numerical analysis to be able to grasp the mechanism of bifurcation, and is numerically efficient and stable enough to put the tangent stiffness matrix into a block-diagonal form. The application of the group-theoretic method to structures, however, was mainly on those with geometrical symmetry.

As a first step, to apply this method to structures with hidden symmetry, the bifurcation mechanism of the simply-supported rectangular plate in Fig. 1 was investigated in Ikeda *et al.* (1996) with reference to the hidden dihedral group symmetry. This study, however, remained incomplete in favor of its simplicity, because it referred only to in-plane symmetry. In this paper, the spatial symmetry of the plate is fully exploited and it is shown to be $D_{\infty h}$-invariant due to the hidden symmetry. The governing equation of this plate is discretized by means of the Galerkin method with the use of the double Fourier series as shape functions. The lattice of subgroups for the plate is obtained. This lattice is quite different from that for a $D_{\infty h}$-equivariant system with geometrical symmetry in that a number of subgroups are absent due to the degeneration by the boundaries. The tangential stiffness matrix of the plate is shown to be block-diagonalized by permuting the order of the Fourier series. The bifurcation analysis of the plate is carried out to assess the validity of the present theory.

2. GROUP-THEORETIC BIFURCATION THEORY

In this section, the group-theoretic bifurcation theory for describing the bifurcation behavior of a system with geometrical symmetry is briefly reviewed and is applied to some specific groups that label spatial symmetries. Necessary background can be found [for example in Sattinger (1979, 1983); Golubitsky and Schaeffer (1985); Golubitsky *et al.* (1988); Dinkevich (1991)].

2.1. *Formulation*

Consider a set of N-dimensional equations

$$\mathbf{F}(\mathbf{u}, f) = \mathbf{0} \tag{1}$$

or, alternatively, a system of governing equation

$$F(w,f) = 0. \tag{2}$$

Here f stands for a loading parameter, $\mathbf{u}$ for a displacement vector, w for a scalar function $w(x, y)$, and (x, y) for the coordinate of a point in the domain; $\mathbf{F}$ and F are assumed to be sufficiently smooth. Although we focus mainly on the discretized form of eqn (1) in this section, its extension to the continuous form of eqn (2) is not difficult (e.g. Ikeda *et al.*, 1994).

Consider a group, G, made up of a series of geometric transformation g, such as reflections and rotations, in describing the symmetry of the equilibrium equation. For example, an element g of a group G transforms an N-dimensional vector $\mathbf{u}$ into $g(\mathbf{u})$. The mechanism of such transformation can be defined by an $N \times N$ representation matrix $T(g)$, which is assumed to be unitary, such that

$$T(g)\mathbf{u} = g(\mathbf{u}), \quad T(g)\mathbf{F} = g(\mathbf{F}), \quad g \in G.$$

The equilibrium equation is said to be equivariant to a group G when

$$T(g)\mathbf{F}(\mathbf{u},f) = \mathbf{F}(T(g)\mathbf{u},f), \quad {}^{\forall}g \in G, \tag{3}$$

is satisfied. The invariance of the solution $\mathbf{u}$, which is a different concept from the equivariance, is defined by $T(g)\mathbf{u} = \mathbf{u}$ $(g \in G)$.

Let $(\mathbf{u}^c, f^c)$ be a critical point on the main path, at which the tangential stiffness matrix (Jacobian) $J(\mathbf{u}^c, f^c) = (\partial \mathbf{F}/\partial \mathbf{u})^c$ has a zero eigenvalue(s). A critical point is either a limit point of the loading parameter f or a bifurcation point. The bifurcation paths, branching from the main path at a bifurcation point, are usually made up of solutions $(\mathbf{u}, f)$ of reduced symmetries, that are to be labeled by subgroups of G.

The multiplicity of a critical point $(\mathbf{u}^c, f^c)$ is defined as

$$M = \dim[\ker(J(\mathbf{u}^c, f^c))],$$

(where ker(·) denotes the kernel space of the linear operator in the parentheses). Let $(\mathbf{u}^c, f^c)$ be a group-theoretic critical point.† Then, by definition, the kernel space X is associated with an irreducible representation μ, which, in turn, is associated with a subgroup $G^{(\mu)}$ of G. To be more specific, let $T^{(\mu)}(g)(g \in G)$ be the representation matrices of the irreducible representation μ. Then

$$G^{(\mu)} = \{g \in G \mid T^{(\mu)}(g) = I_M\}, \tag{4}$$

where I_M is the $M \times M$ identity matrix. Note that the degree of the irreducible representation is equal to the multiplicity M of the critical point.

2.2. *Bifurcation equation*

The elimination of the passive coordinates [see Thompson and Hunt (1973)], which is called the Liapunov–Schmidt decomposition for an infinite-dimensional system, reduces the original equation to a system of (bifurcation) equations. Let $\mathbf{e}_i^c$ $(i = 1, \ldots, M)$ be M independent eigenvectors (eigenfunctions) of the zero eigenvalue at a bifurcation point $(\mathbf{u}^c, f^c)$ of multiplicity M. Then we can define incremental variables $\tilde{\mathbf{w}} = (\tilde{w}_1, \ldots, \tilde{w}_M)^T$ and $\tilde{f}$ from the critical point $(\mathbf{u}^c, f^c)$ as:

$$\mathbf{u} = \sum_{i=1}^{M} \tilde{w}_i \mathbf{e}_i^c + \mathbf{u}^c, \quad f = f^c + \tilde{f}.$$

Then the bifurcation equation reads

† The critical point is divided into two types, group-theoretic or parametric, according to whether X is G-irreducible or not. Since the latter is rare, we consider only the former type in the remainder of this paper.

$$\tilde{\mathbf{F}}(\tilde{\mathbf{w}},\tilde{f}) = \mathbf{0}. \tag{5}$$

When the original equation $\mathbf{F}$ is equivariant to G, the bifurcation equation $\tilde{\mathbf{F}}$ can be chosen to be equivariant to G with respect to the associated irreducible representation, that is,

$$\tilde{T}(g)\tilde{\mathbf{F}}(\tilde{\mathbf{w}},\tilde{f}) = \tilde{\mathbf{F}}(\tilde{T}(g)\tilde{\mathbf{w}},\tilde{f}), \quad {}^{\forall}g \in G, \tag{6}$$

where $\tilde{T}(g)$ is the $M \times M$ irreducible representation matrix. It is this inheritance of symmetry to the bifurcation equation that plays a key role in determining possible bifurcating solutions and their symmetries at a critical point (cf. Appendix). Note that the symmetry of these bifurcating solutions is often higher than that of the kernel space $G^{(\mu)}$, which also labels the symmetry of the eigenvectors $\mathbf{e}_i^c$.

2.3. *Block-diagonalization*

For a G-invariant $\mathbf{u}$, the tangential stiffness matrix J satisfies a symmetry condition

$$T(g)J = JT(g), \quad {}^{\forall}g \in G$$

by eqn (3) and, hence, can be put into a block-diagonal form by means of a suitable transformation matrix H. The form of H, which depends on the definition of $\mathbf{u}$ and G, will be given later in this paper for particular cases.

The space X for $\mathbf{u}$ can be decomposed into the direct sum of subspaces X^{μ}

$$X = \bigoplus_{\mu \in R(G)} X^{\mu} \tag{7}$$

by means of the isotypic (standard) decomposition, where $\oplus$ denotes the direct sum and $R(G)$ denotes the whole set of irreducible representations. Each subspace is associated with the solution for the main or a bifurcation path. It is to be noted that the space X^{μ} that corresponds to an M-dimensional irreducible representation can be further decomposed into the direct sum of M subspaces, that is,

$$X^{\mu} = \bigoplus_{i=1}^{M} X^{\mu^i}. \tag{8}$$

The transformation matrix can also be decomposed into the form of

$$H \equiv [\ldots, H^{\mu}, \ldots]$$

made up of blocks H^{μ} associated with the irreducible representations. With the use of this H the tangential stiffness matrix can be block-diagonalized, that is,

$$\tilde{J}^{G} = (H^{G})^{T} J H^{G} = \mathrm{diag}[\ldots, \tilde{J}^{\mu}, \ldots], \quad \mu \in R(G), \tag{9}$$

where diag[...] denotes a block-diagonal matrix. It is to be noted that a block $\tilde{J}^{\mu}$ that corresponds to an M-dimensional irreducible representation can be further decomposed into a block-diagonal form with M identical blocks, that is,

$$\tilde{J}^{\mu} = \mathrm{diag}[\hat{J}^{\mu}, \ldots, \hat{J}^{\mu}], \quad \mu \in R(G) \tag{10}$$

to be consistent with eqn (8). The multiplicity of the zero eigenvalues of $\tilde{J}^{\mu}$ corresponding to an M-dimensional irreducible representation is necessarily repeated M times. The block-diagonalization method is advantageous in that the singularity of J is distributed into a few blocks and that the block-diagonal form of eqn (9) corresponds to the categorization of singular points (e.g. Murota and Ikeda (1991); Ikeda and Murota (1991)].

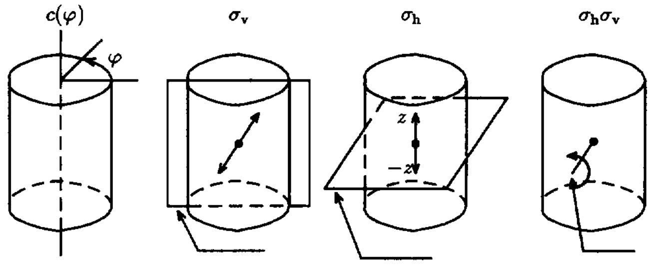

Fig. 2. Actions of the elements σ_h, σ_v, $c(\varphi)$ and $\sigma_h\sigma_v$.

2.4. *Schoenflies notation for describing symmetries*

We introduce the Schoenflies notation for describing geometric symmetries. [See, e.g. Kettle (1995) for details of this notation.] As shown in Fig. 2, consider the reflection σ_v with respect to a vertical plane, the reflection

$$\sigma_h : z \rightarrow -z$$

with respect to the horizontal xy-plane, and the rotation

$$c(\varphi) : \theta \rightarrow \theta + \varphi \tag{11}$$

around the z-axis at an angle of $\varphi (0 \leqslant \varphi \leqslant 2\pi)$, where $\theta = \tan^{-1}(y/x)$. Then the symmetry of the system invariant under these three transformations is labeled by

$$D_{\infty h} = \langle \sigma_v, \sigma_h, c(\varphi) \rangle, \quad 0 \leqslant \varphi < 2\pi,$$

where the brackets $\langle \cdot \rangle$ denote the group generated by the elements therein.

The subgroups of $D_{\infty h}$ are [e.g. Dinkevich (1991)] :†

$$\begin{aligned}
D_{nh} &= \langle \sigma_v, \sigma_h, c(2\pi/n) \rangle, \\
D_{nd} &= \langle \sigma_h \sigma_v, \sigma_v\, c(\pi/n) \rangle, \\
C_{\infty v} &= \langle \sigma_v, c(\varphi) \rangle, \quad C_{nv} = \langle \sigma_v, c(2\pi/n) \rangle, \\
C_{\infty h} &= \langle \sigma_h, c(\varphi) \rangle, \quad C_{nh} = \langle \sigma_h, c(2\pi/n) \rangle, \\
D_{\infty} &= \langle \sigma_h \sigma_v, c(\varphi) \rangle, \quad D_n = \langle \sigma_h \sigma_v, c(2\pi/n) \rangle, \\
S_n &= \langle \sigma_h c(2\pi/n) \rangle, \\
C_{\infty} &= \langle c(\varphi) \rangle, \quad C_n = \langle c(2\pi/n) \rangle,
\end{aligned}$$

where n is an integer denoting the frequency of rotational symmetry and the subscript, ∞, denotes the axisymmetry. The action of $\sigma_h\sigma_v$, for example, is shown in Fig. 2.

The group D_{nd}, which play a key role in this paper, is expressed alternatively as:

$$\begin{aligned}
D_{nd} = \{ & e, c(\pi 2/n), \ldots, c(\pi 2(n-1)/n), \\
& \sigma_h \sigma_v, \sigma_h \sigma_v c(\pi 2/n), \ldots, \sigma_h \sigma_v c(\pi 2(n-1)/n), \\
& \sigma_h c(\pi/n), \sigma_h c(\pi 3/n), \ldots, \sigma_h c(\pi(2n-1)/n), \\
& \sigma_v c(\pi/n), \sigma_v c(\pi 3/n), \ldots, \sigma_v c(\pi(2n-1)/n) \},
\end{aligned}$$

† It should be noted that in the context of the Schoenflies notation the definition of σ_v is not unique, and σ_v and $\sigma_v c(\varphi)$ are sometimes identified.

where e is the identity element that leaves everything unchanged; the braces $\{\cdot\}$ denote the elements of the group; and the product of elements denotes that the transformations are performed from the right to the left in sequence.

2.5. $D_{\infty h}$-equivariant system

We offer here the rule for the direct bifurcation of a $D_{\infty h}$-equivariant system. We index the family of nonequivalent irreducible representations of $D_{\infty h}$ by

$$R(D_{\infty h}) = \{(\nu_1, \nu_2)_{D_{\infty h}}, (n, \nu_2)_{D_{\infty h}} \mid \nu_1, \nu_2 = +, -, n = 1, 2, \ldots\}. \tag{12}$$

Here $(+,+)_{D_{\infty h}}$ corresponds to the unit one-dimensional irreducible representation and $(+,-)_{D_{\infty h}}$, $(-,+)_{D_{\infty h}}$ and $(-,-)_{D_{\infty h}}$ to the remaining one-dimensional irreducible representations, which are defined by the one-dimensional representation matrices $T^{(\mu)}(\cdot)$:

$$\begin{aligned}
&T^{(+,+)_{D_{\infty h}}}(\sigma_v) = 1, \quad T^{(+,+)_{D_{\infty h}}}(\sigma_h) = 1, \quad T^{(+,+)_{D_{\infty h}}}(c(\varphi)) = 1,\\
&T^{(+,-)_{D_{\infty h}}}(\sigma_v) = 1, \quad T^{(+,-)_{D_{\infty h}}}(\sigma_h) = -1, \quad T^{(+,-)_{D_{\infty h}}}(c(\varphi)) = 1,\\
&T^{(-,+)_{D_{\infty h}}}(\sigma_v) = -1, \quad T^{(-,+)_{D_{\infty h}}}(\sigma_h) = 1, \quad T^{(-,+)_{D_{\infty h}}}(c(\varphi)) = 1,\\
&T^{(-,-)_{D_{\infty h}}}(\sigma_v) = -1, \quad T^{(-,-)_{D_{\infty h}}}(\sigma_h) = -1, \quad T^{(-,-)_{D_{\infty h}}}(c(\varphi)) = 1.
\end{aligned} \tag{13}$$

By eqn (4), the symmetry groups $G^{(\mu)}$ of the kernel spaces for the one-dimensional irreducible representations are

$$G^{(+,+)_{D_{\infty h}}} = D_{\infty h}, \quad G^{(+,-)_{D_{\infty h}}} = C_{\infty v}, \quad G^{(-,+)_{D_{\infty h}}} = C_{\infty h}, \quad G^{(-,-)_{D_{\infty h}}} = D_{\infty}.$$

The one-dimensional unit representation $(+,+)_{D_{\infty h}}$ is associated with a limit point of the loading parameter f, and its symmetry is labeled by the group $D_{\infty h}$. The one-dimensional nonunit representations $(+,-)_{D_{\infty h}}$, $(-,+)_{D_{\infty h}}$, and $(-,-)_{D_{\infty h}}$ are associated with simple, symmetric bifurcation points with $C_{\infty v}$-, $C_{\infty h}$- and D_{∞}- invariant bifurcation modes, respectively.

The remaining irreducible representations in eqn (12) are of degree two, and their actions are defined by

$$\begin{aligned}
&T^{(n,+)_{D_{\infty h}}}(\sigma_v) = P, \quad T^{(n,+)_{D_{\infty h}}}(\sigma_h) = I_2, \quad T^{(n,+)_{D_{\infty h}}}(c(\varphi)) = R,\\
&T^{(n,-)_{D_{\infty h}}}(\sigma_v) = P, \quad T^{(n,-)_{D_{\infty h}}}(\sigma_h) = -I_2, \quad T^{(n,-)_{D_{\infty h}}}(c(\varphi)) = R,
\end{aligned} \tag{14}$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{pmatrix}.$$

The two-dimensional irreducible representations $(n,+)_{D_{\infty h}}$ and $(n,-)_{D_{\infty h}}$ are associated with symmetric double bifurcation points, from which D_{nh}- and D_{nd}-invariant bifurcation paths,

Table 1. Representation matrices and symmetry groups of bifurcation points and paths of $D_{\infty h}$-invariant system

Multiplicity M	Irreducible rep. μ	$T^{(\mu)}(g)$ σ_v	σ_h	$c(\varphi)$	Symmetry groups $G^{(\mu)}$ for X	Bifurcation paths
1	$(+,+)_{D_{\infty h}}$	1	1	1	$D_{\infty h}$	$D_{\infty h}$
	$(+,-)_{D_{\infty h}}$	1	−1	1	$C_{\infty v}$	$C_{\infty v}$
	$(-,+)_{D_{\infty h}}$	−1	1	1	$C_{\infty h}$	$C_{\infty h}$
	$(-,-)_{D_{\infty h}}$	−1	−1	1	D_{∞}	D_{∞}
2	$(n,+)_{D_{\infty h}}$	P	I_2	R	C_{nh}	D_{nh}
	$(n,-)_{D_{\infty h}}$	P	$-I_2$	R	S_{2n}	D_{nd}

Here $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $R = \begin{pmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{pmatrix}$.

respectively, branch. The symmetries of the kernel space and the bifurcation paths are summarized in Table 1.

2.6. D_{nd}-equivariant system

The rule of the direct bifurcation of a D_{nd}-equivariant system is obtained. We index the family of nonequivalent irreducible representations of D_{nd} by

$$R(D_{nd}) = \{(+,+)_{D_{nd}},(+,-)_{D_{nd}},(-,+)_{D_{nd}},(-,-)_{D_{nd}},(1)_{D_{nd}},(2)_{D_{nd}},\ldots,(n-1)_{D_{nd}}\}. \tag{15}$$

Here $(+,+)_{D_{nd}}$ corresponds to the unit (one-dimensional) irreducible representation and $(+,-)_{D_{nd}}$, $(-,+)_{D_{nd}}$, and $(-,-)_{D_{nd}}$ to other one-dimensional ones, which are defined by the one-dimensional representation matrices in Table 2. The irreducible representations $(1)_{D_{nd}}, (2)_{D_{nd}}, \ldots,$ and $(n-1)_{D_{nd}}$ in eqn (15) are of degree two, and their actions are also defined in this table.

For simple critical points, associated with one-dimensional irreducible representations, we can see from Table 2 that the kernel space is labeled by the groups

$$G^{(+,+)_{D_{nd}}} = D_{nd}, \quad G^{(+,-)_{D_{nd}}} = S_{2n}, \quad G^{(-,+)_{D_{nd}}} = D_n, \quad G^{(-,-)_{D_{nd}}} = C_{nv},$$

which also represent the symmetries of the (bifurcation) paths. The group D_{nd} is associated with a limit point of a loading parameter f, while the subgroups S_{2n}, D_n and C_{nv} to simple, symmetric bifurcation points.

The solutions of the two-dimensional bifurcation equation (5) of a double point ($M = 2$) associated with the jth two-dimensional irreducible representation $(j)_{D_{nd}}$ of the D_{nd}-equivariant system are obtained in the Appendix. To sum up, the symmetries of the bifurcation paths are labeled by

Table 2. Representation matrices and symmetry groups of bifurcation points and paths of D_{nd}-invariant system

Multiplicity M	Irreducible rep. μ	$T^{(\mu)}(g)$ $\sigma_h\sigma_v$	$\sigma_h c(\pi/n)$	Symmetry groups $G^{(\mu)}$	Bifurcation paths
1	$(+,+)_{D_{nd}}$	1	1	D_{nd}	D_{nd}
	$(+,-)_{D_{nd}}$	−1	1	S_{2n}	S_{2n}
	$(-,+)_{D_{nd}}$	1	−1	D_n	D_n
	$(-,-)_{D_{nd}}$	−1	−1	C_{nv}	C_{nv}
2	$(j)_{D_{nd}}$	P	Q	$S_{2n/\hat{n}}$ ($\hat{j}+\hat{n}$ is even) $C_{n/\hat{n}}$ ($\hat{j}+\hat{n}$ is odd)	$D_{(n/\hat{n})d}$ $C_{(n/\hat{n})v}$ or $D_{n/\hat{n}}$

Here $Q = -\begin{bmatrix} \cos\frac{\pi j}{n} & -\sin\frac{\pi j}{n} \\ \sin\frac{\pi j}{n} & \cos\frac{\pi j}{n} \end{bmatrix}$, $\hat{j} = \frac{j}{\gcd(j,n)}$, $\hat{n} = \frac{n}{\gcd(j,n)}$, and $\gcd(j,n)$ denotes the greatest common divisor of j and n.

Table 3. Representation matrices and symmetry groups of bifurcation points and paths of D_n-invariant system

Multiplicity M	Irreducible rep. μ	$T^{(\mu)}(g)$ $\sigma_h\sigma_v$	$c(\pi 2/n)$	Symmetry groups $G^{(\mu)}$	Bifurcation paths
1	$(+,+)_{D_n}$	1	1	D_n	D_n
	$(+,-)_{D_n}$	-1	1	C_n	C_n
	$(-,+)_{D_n}$	1	-1	$D_{n/2}$	$D_{n/2}$
	$(-,-)_{D_n}$	-1	-1	$D_{n/2}$	$D_{n/2}$
2	$(j)_{D_n}$	$-P$	Q^2	$C_{n/\hat{n}}$	$D_{n/\hat{n}}$

$$\begin{cases} D_{(n/\hat{n})d} & \text{if } \hat{j}+\hat{n} \text{ is even,} \\ C_{(n/\hat{n})v} \quad \text{or} \quad D_{n/\hat{n}} & \text{if } \hat{j}+\hat{n} \text{ is odd,} \end{cases}$$

where

$$\hat{j} = \frac{j}{\gcd(j,n)}, \quad \hat{n} = \frac{n}{\gcd(j,n)},$$

and $\gcd(j, n)$ is the greatest common divisor of j and n. This point is asymmetric for $\hat{j}+\hat{n}$ is even and symmetric when it is odd.

2.7. *D_n-equivariant system*

We index the family of nonequivalent irreducible representations of D_n by

$$R(D_n) = \begin{cases} \{(+,+)_{D_n},(+,-)_{D_n},(1)_{D_n},(2)_{D_n},\ldots,((n-1)/2)_{D_n}\} & \text{if } n \text{ is odd,} \\ \{(+,+)_{D_n},(+,-)_{D_n},(-,+)_{D_n}(-,-)_{D_n}, & \\ \quad (1)_{D_n},(2)_{D_n},\ldots,(n/2-1)_{D_n}\} & \text{if } n \text{ is even.} \end{cases}$$

Here $(+,+)_{D_n}$ corresponds to the unit (one-dimensional) irreducible representation and $(+,-)_{D_n}$, $(-,+)_{D_n}$ and $(-,-)_{D_n}$ to other one-dimensional ones, and the remaining irreducible representations are of degree two. The actions of the representation matrices are given in Table 3.

For simple critical points associated with the one-dimensional ones, we can see from Table 3 that the symmetries of the kernel spaces and, hence, the symmetries of the (bifurcation) paths, are labeled by

$$G^{(+,+)_{D_n}} = D_n, \quad G^{(+,-)_{D_n}} = C_n, \quad G^{(-,+)_{D_n}} = D_{n/2}, \quad G^{(-,-)_{D_n}} = D_{n/2}.$$

The group D_n is associated with a limit point of the loading parameter f, while the subgroups C_n and $D_{n/2}$ to simple, symmetric bifurcation points.

Because D_n is isomorphic to C_{nv}, the bifurcation structure of the D_n-equivariant system is identical to that of C_{nv}-equivariant one [cf. Ikeda *et al.* (1991)]. To sum up, the double bifurcation point associated with the two-dimensional irreducible representation $(j)_{D_n}$ is asymmetric when $\hat{n}$ is odd and symmetric when even, and the symmetry of the system on all bifurcation paths is labeled by $D_{n/\hat{n}}$.

2.8. *Bifurcation hierarchy*

The rules of further bifurcation from the subgroups of $D_{\infty h}$, in addition to the groups D_{nd} and D_n, are to be obtained in a similar manner. The assemblage of the rules for $D_{\infty h}$

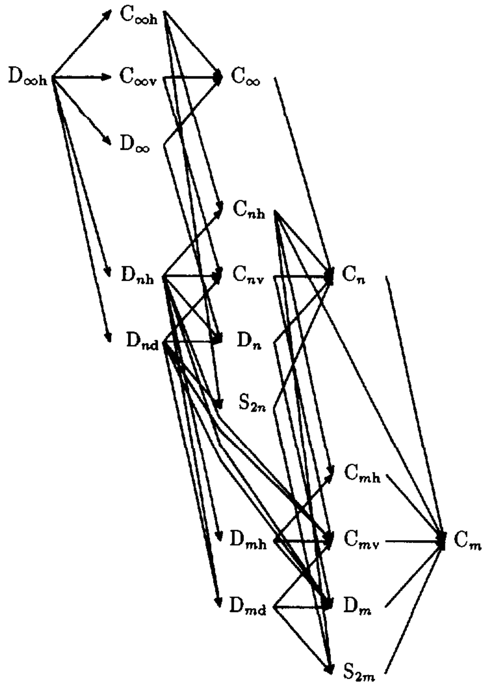

Fig. 3. Bifurcation among subgroups of $D_{\infty h}$: a lattice of subgroups expressing the bifurcation rule (n/m is odd for $D_{nd} \rightarrow D_{md}$; n/m is even for $D_{nh} \rightarrow D_{md}$ and $C_{nh} \rightarrow S_{2m}$).

and its subgroups leads to the bifurcation rule in Fig. 3 (Ikeda and Murota, 1997), which is expressed in terms of a lattice of subgroups.

3. RECTANGULAR PLATE

The governing equation of a four-sides simply-supported rectangular plate subject to a pair of uniaxial pure bending moments in Fig. 1 is presented and its symmetry is described.

3.1. *Governing equation*

We employ the nonlinear differential equation by von Kármán:

$$\nabla^4 w = \frac{t}{D}\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right), \tag{16}$$

$$\nabla^4 \phi = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right], \tag{17}$$

$(0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b)$, where $w(x, y)$ denotes the out-of-plane deflection; $\phi(x, y)$ indicates the in-plane stress function; t and $D \equiv Et^3/[12(1-\nu^2)]$ are the plate thickness and the flexural rigidity of the plate, respectively; E is Young's modulus and ν is Poisson's ratio.

The in-plane stress components are expressed in terms of the Airy stress function $\phi(x, y)$ as

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

The mechanical boundary conditions are expressed as

$$\begin{cases} M = -t\displaystyle\int_0^b \sigma_x(x,y)\left(y-\frac{b}{2}\right)\mathrm{d}y, \quad \int_0^b \sigma_x\,\mathrm{d}y = 0, \quad \tau_{xy} = 0 \quad \text{for } x = 0, a, \\ \displaystyle\int_0^a \sigma_y\,\mathrm{d}x = 0, \quad \tau_{xy} = 0 \qquad \text{for } y = 0, b. \end{cases} \tag{18}$$

The boundary conditions in eqn (18) can be exactly satisfied if a homogeneous solution of eqn (17):

$$\phi_0(y) = -\frac{y^2(2y-3b)}{b^3 t}M, \tag{19}$$

is chosen to be the Airy stress function (Nakazawa *et al.*, 1991).

The equilibrium equation can, consequently, be expressed as

$$F(w(x,y),f) = 0, \tag{20}$$

where $(w(x,y),f)$ satisfies the governing eqns (16) and (17) and the boundary conditions in eqn (18). In this equation, the loading parameter is defined as

$$f \equiv \frac{M}{D}.$$

3.2. *Discretization by Galerkin's method*

Galerkin's method is applied to eqn (20) to arrive at a discretized form of the governing equation. For the four-sides simply-supported boundary condition, the deflection w can be approximated by the double Fourier expansion, that is,

$$w = t\sum_{i=1}^{m_x}\sum_{j=1}^{m_y} u_{ij}\sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right), \quad 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b, \tag{21}$$

where m_x and m_y denote the numbers of the Fourier series in the x- and y-directions, respectively. Define the displacement vector $\mathbf{u}$ by

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u}_{m_x} \end{bmatrix}, \quad \mathbf{u}_i = \begin{bmatrix} u_{i1} \\ \cdot \\ \cdot \\ \cdot \\ u_{im_y} \end{bmatrix}, \quad \begin{matrix} i = 1,\ldots,m_x, \\ j = 1,\ldots,m_y. \end{matrix} \tag{22}$$

Then the governing equation (20) can be rewritten in a discretized form as:

$$\mathbf{F}(\mathbf{u},f) = \mathbf{0}, \tag{23}$$

where the components of the vector $\mathbf{F}$ are expressed as

$$F_{(i-1)m_y+j}(\mathbf{u},f) \equiv \int_0^a\int_0^b \left[\nabla^4 w - \frac{t}{D}\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right)\right]$$
$$\cdot\sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right)\mathrm{d}x\,\mathrm{d}y, \quad i = 1,\ldots,m_x, j = 1,\ldots,m_y. \tag{24}$$

The general expression of $\phi(x, y)$ is obtained by summing the homogeneous solution $\phi_0(y)$ of eqn (19) and a particular solution of the right-hand side of eqn (17), that is,

$$\phi(x,y) = \phi_0(y) + Et^2 \sum_{p=0}^{m_x} \sum_{q=0}^{m_y} \phi_{pq} \cos\left(\frac{p\pi x}{a}\right) \cos\left(\frac{2q\pi y}{b}\right). \tag{25}$$

Substituting eqns (21) and (25) into eqn (17), we obtain the expression of ϕ_{pq} in terms of u_{ij} as

$$\frac{8(p^2+4\alpha^2 q^2)^2}{\alpha^2} \phi_{pq} = \sum_{m=1}^{m_x} \sum_{n=1}^{m_y} \sum_{i=1}^{m_x} \sum_{j=1}^{m_y} u_{mn} u_{ij} [2mnij \pm (m^2 j^2 + n^2 i^2)], \tag{26}$$

where $\alpha \equiv a/b$ is the aspect ratio of a panel, and p and q are positive integers, being expressed as

$$- : \begin{cases} p = m+i \quad \text{and} \quad q = (n+j)/2 \\ \text{or} \\ p = |m-i| \quad \text{and} \quad q = |n-j|/2 \end{cases} \qquad + : \begin{cases} p = |m-i| \quad \text{and} \quad q = (n+j)/2 \\ \text{or} \\ p = m+i \quad \text{and} \quad q = |n-j|/2 \end{cases}$$

according to whether $\pm$ in eqn (26) is $-$ or $+$.

By substituting eqns (21) and (25) into eqn (23) and integrating it, we obtain a set of third-order simultaneous algebraic equations of u_{ij}s. The substitution of the obtained u_{ij}s into eqn (21) yields the out-of-plane deflection in the post-buckling state.

3.3. *Description of symmetry*

The symmetries of the initial and deformed states of the plate in Fig. 1 are labeled here with the use of the Schoenflies notation. In order to exploit the periodic symmetry of the plate, we extend the domain in eqns (16) and (17) two-fold in the x-direction, that is,

$$0 \leqslant x \leqslant a \rightarrow -a \leqslant x \leqslant a.$$

Then it is easy to see that $F(w(x,y),f)$ in eqn (20) is equivariant under the action of

$$\sigma_h \sigma_v : z \rightarrow -z \quad (w \rightarrow -w) \quad \text{and} \quad x \rightarrow -x,$$

that is,

$$F(-w(-x,y),f) = -F(w(x,y),f) \tag{27}$$

is satisfied. It indicates that if $(w(x,y),f)$ $(0 \leqslant x \leqslant a)$ is a solution of $F(w(x,y),f) = 0$ in eqn (20), $(-w(-x,y),f)$ $(-a \leqslant x \leqslant 0)$ is also a solution. Hence, the Fourier expansion of eqn (21), which satisfies

$$-w(-x,y) = w(x,y)$$

serves as a solution for the extended domain. Since w in this expansion is periodic at the ends $x = \pm a$, the solution w of the plate extended in this manner is periodic at the ends.

The geometry of the extended plate in the initial state is invariant under the action of the horizontal reflection σ_h with respect to the xy-plane (due to its upside-down symmetry), that of the reflection σ_v with respect to the vertical yz-plane, and of the translation

$$c(\varphi) : x \rightarrow x + l$$

in the x-direction at a length of l. This translation, which is due to the periodic nature of

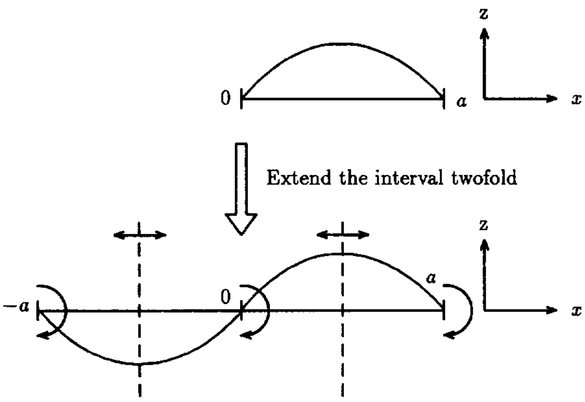

Fig. 4. Schematic illustration of the D_{1d}-invariance of $\sin(\pi x/a)$. The dashed lines (---) denote the planes of reflection symmetry.

w, is expressed by $c(\varphi)$ by setting $l = a\varphi/\pi(-a \leqslant l \leqslant a, 0 \leqslant \varphi \leqslant 2\pi)$ in eqn (11). The symmetry of the extended plate, accordingly, is labeled by the group

$$D_{\infty h} = \langle \sigma_v, \sigma_h, c(\varphi) \rangle, \quad 0 \leqslant \varphi < 2\pi.$$

It is easy to see that the approximation by Galerkin's method does not break the symmetry of the system, that is, eqn (24) retains $D_{\infty h}$-equivariance. Of course, the finiteness of the terms in eqn (24) takes effect, and the system discretized in this manner loses some bifurcation modes.

The symmetry of the nth Fourier coefficients in the x-direction

$$\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right), \quad j = 1, 2, \ldots$$

is labeled by the group D_{nd}. Figure 4 schematically illustrates the D_{1d}-invariance of $\sin(\pi x/a)$ in the extended interval $-a \leqslant x \leqslant a$. In this figure, the axes of half rotation symmetry are located at $x = 0$, $\pm a$, and the planes of reflection symmetry at $x = \pm a/2$.

It is to be emphasized that the displacement w in eqn (21) expressed in terms of the double Fourier series is always invariant under the action of $\sigma_h\sigma_v$ and, hence, the displacement after bifurcation is also invariant under either D_{nd} or D_n for some integer n. The simply-supported boundary condition thus restricts the symmetry of the bifurcated state.

4. BIFURCATION RULE

The bifurcation rule of the rectangular plate in Fig. 1 is presented in this section. This rule suffers degeneration owing to the restriction by the boundary conditions, compared with those for the $D_{\infty h}$-equivariant system presented in Section 2. In particular, block-diagonal forms of the tangential stiffness matrix are obtained for $D_{\infty h}$-, D_{nd}-, and D_n-invariant paths of the plate. The numbers m_x and m_y of the double Fourier expansion in eqn (21) are chosen to be ∞ to simplify the resulting formulas.

4.1. $D_{\infty h}$-invariant main path

We consider the space $\hat{X}$ of w in eqn (21) that satisfies the boundary conditions, and the space X of w that does not. These spaces are spanned, respectively, by

$$\hat{X} = \text{span}\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{n,l=1}^{\infty}$$

$$X = \text{span}\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right), \cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{n,l=1}^{\infty}$$

$(-a \leqslant x \leqslant a)$. Here span[·] indicates that the relevant space is spanned by the functions therein. The structure of bifurcation for the space X has already been presented in Section 2. In this section, we investigate the way this mechanism is inherited to the space $\hat{X}$, which is restricted by the boundary conditions, to obtain the Fourier series spanning the subspaces $\hat{X}^{\mu}$ of $\hat{X}$.

The isotypic decomposition (7) of the space X of w with respect to the group $\mathrm{D}_{\infty\mathrm{h}}$ is expressed as:

$$X = X^{(+,+)_{\mathrm{D}_{\infty\mathrm{h}}}} \oplus X^{(+,-)_{\mathrm{D}_{\infty\mathrm{h}}}} \oplus X^{(-,+)_{\mathrm{D}_{\infty\mathrm{h}}}} \oplus X^{(-,-)_{\mathrm{D}_{\infty\mathrm{h}}}} \oplus \left[\bigoplus_{n=1}^{\infty}(X^{(n,+)_{\mathrm{D}_{\infty\mathrm{h}}}} \oplus X^{(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}})\right] \quad (28)$$

to be consistent with the composition of the irreducible representations in eqn (12). Furthermore, the subspaces $X^{(n,+)_{\mathrm{D}_{\infty\mathrm{h}}}}$ and $X^{(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}}$ for two-dimensional irreducible representations are further decomposed into:

$$X^{(n,+)_{\mathrm{D}_{\infty\mathrm{h}}}} = X^{(n,+)^1_{\mathrm{D}_{\infty\mathrm{h}}}} \oplus X^{(n,+)^2_{\mathrm{D}_{\infty\mathrm{h}}}}, \quad X^{(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}} = X^{(n,-)^1_{\mathrm{D}_{\infty\mathrm{h}}}} \oplus X^{(n,-)^2_{\mathrm{D}_{\infty\mathrm{h}}}}. \quad (29)$$

The Fourier series spanning each subspace is determined with reference to eqns (13) and (14), and to the actions of σ_{h}, σ_{v} and $c(\varphi)$ defined as shown in Fig. 2. For example, for $(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}$, the action of σ_{h} on the Fourier series

$$\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right) \quad \text{and} \quad \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right) \quad (30)$$

satisfies

$$\sigma_{\mathrm{h}} \cdot \mathbf{q} = -\mathbf{q} = -I_2\mathbf{q} = T^{(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}}(\sigma_{\mathrm{h}})\mathbf{q}$$

by eqn (14), where

$$\mathbf{q} = \left(\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right), \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right)^{\mathrm{T}}.$$

In addition, the actions of σ_{v} and $c(\varphi)$, respectively, satisfy

$$\sigma_{\mathrm{v}} \cdot \mathbf{q} = T^{(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}}(\sigma_{\mathrm{v}})\mathbf{q}, \quad c(\varphi) \cdot \mathbf{q} = T^{(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}}(c(\varphi))\mathbf{q}.$$

The Fourier series in eqn (30), accordingly, correspond to the two-dimensional irreducible representation $(n,-)_{\mathrm{D}_{\infty\mathrm{h}}}$. The subspaces in eqns (28) and (29), therefore, are spanned by:

$$X^{(+,+)_{\mathrm{D}_{\infty\mathrm{h}}}} = \text{span}[0],$$

$$X^{(+,-)_{\mathrm{D}_{\infty\mathrm{h}}}} = X^{(-,+)_{\mathrm{D}_{\infty\mathrm{h}}}} = X^{(-,-)_{\mathrm{D}_{\infty\mathrm{h}}}} = X^{(n,+)_{\mathrm{D}_{\infty\mathrm{h}}}} = \text{span}[\phi],$$

$$X^{(n,-)^1_{\mathrm{D}_{\infty\mathrm{h}}}} = \text{span}\left[\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{l=1}^{\infty}.$$

$$X^{(n,-)^2_{\mathrm{D}_{\infty\mathrm{h}}}} = \text{span}\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{l=1}^{\infty}, \quad (31)$$

where ϕ denotes that the relevant space does not exist for this case.

In view of the fact that the cosine terms of x do not exist for the space, $\hat{X}$, the space $\hat{X}$ is decomposed as follows:

$$\hat{X} = \hat{X}^{(+,+)_{D_{\infty h}}} \oplus \left(\bigoplus_{n=1}^{\infty} \hat{X}^{(n,-)_{D_{\infty h}}} \right),$$

where

$$\hat{X}^{(+,+)_{D_{\infty h}}} = X^{(+,+)_{D_{\infty h}}}, \quad \hat{X}^{(n,-)_{D_{\infty h}}} = X^{(n,-)^2_{D_{\infty h}}}. \tag{32}$$

By eqns (31) and (32), we can see that the subspaces of $\hat{X}$ are spanned by

$$\begin{aligned} \hat{X}^{(+,+)_{D_{\infty h}}} &= \mathrm{span}[0], \\ \hat{X}^{(n,-)_{D_{\infty h}}} &= \mathrm{span}\left[\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right) \right]_{l=1}^{\infty}. \end{aligned} \tag{33}$$

Their symmetries are labeled by

$$\Sigma(\hat{X}^{(+,+)_{D_{\infty h}}}) = D_{\infty h}, \quad \Sigma(\hat{X}^{(n,-)_{D_{\infty h}}}) = D_{nd},$$

where $\Sigma(\cdot)$ indicates the symmetry of the space therein.

Since each displacement component of the vector $\mathbf{u}$ in eqn (22) corresponds to a subspace in eqn (33), the matrix J is already in a block-diagonal form:

$$\tilde{J}^{D_{\infty h}} = J = \mathrm{diag}[\tilde{J}^{(1,-)_{D_{\infty h}}}, \tilde{J}^{(2,-)_{D_{\infty h}}}, \ldots].$$

Furthermore, the diagonal blocks $\tilde{J}^{(n,-)_{D_{\infty h}}}$ $(n = 1, 2, \ldots)$ are all of diagonal forms due to the orthogonality of the double Fourier series. When these blocks are all regular, the system has the trivial $D_{\infty h}$-invariant solution $w = 0$. A diagonal block $\tilde{J}^{(n,-)_{D_{\infty h}}}$ becomes singular at a simple bifurcation point, at which D_{nd}-invariant bifurcation paths branch. This point, which is associated with a two-dimensional irreducible representation, is not a double point, but a simple one because $\tilde{J}^{(n,-)_{D_{\infty h}}}$ cannot be put into the form of eqn (10) due to the degeneration by the boundary conditions. To be precise, although the two blocks in eqn (10) are associated, respectively, with the terms $\sin(n\pi x/a)$ and $\cos(n\pi x/a)$, the latter term and, hence, the block for the latter, is absent due to the boundary condition.

4.2. D_{nd}-invariant bifurcation path

The rule of the bifurcation of the D_{nd}-invariant bifurcation path can be obtained similarly. With respect to the group D_{nd}, the space $\hat{X}$ is decomposed into:

$$\hat{X} = \bigoplus_{\mu \in R(D_{nd})} \hat{X}^{\mu} = \hat{X}^{(+,+)_{D_{nd}}} \oplus \hat{X}^{(-,+)_{D_{nd}}} \oplus \left(\bigoplus_{j=1}^{n-1} \hat{X}^{(j)_{D_{nd}}} \right), \tag{34}$$

In this equation the subspaces of the irreducible representations $(+,-)_{D_{nd}}$ and $(-,-)_{D_{nd}}$ are absent. Each subspace in eqn (34) is spanned by:

$$\hat{X}^{(+,+)_{D_{nd}}} = \text{span}\left[\sin\left((2k-1)n\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{k,l=1}^{\infty},$$

$$\hat{X}^{(-,+)_{D_{nd}}} = \text{span}\left[\sin\left(2kn\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{k,l=1}^{\infty},$$

$$\hat{X}^{(j)_{D_{nd}}} = \text{span}\left[\sin\left([2(k-1)n+j]\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right),\right.$$

$$\left.\sin\left((2kn-j)\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{k,l=1}^{\infty}. \tag{35}$$

The symmetry of each subspace is labeled by

$$\Sigma(\hat{X}^{(+,+)_{D_{nd}}}) = D_{nd}, \quad \Sigma(\hat{X}^{(-,+)_{D_{nd}}}) = D_n,$$

$$\Sigma(\hat{X}^{(j)_{D_{nd}}}) = \begin{cases} D_{(n/\hat{n})d} & \text{if } \hat{j}+\hat{n} \text{ is even,} \\ D_{n/\hat{n}} & \text{if } \hat{j}+\hat{n} \text{ is odd,} \end{cases}$$

where $\hat{j} = j/\gcd(j,n)$ and $\hat{n} = n/\gcd(j,n)$. Here, $\gcd(j,n)$ denotes the greatest common divisor of j and n, and $\hat{j}$ and $\hat{n}$ are relatively prime.

The block-diagonal form of the tangential stiffness matrix becomes

$$\tilde{J}^{D_{nd}} = (H^{D_{nd}})^{T} J H^{D_{nd}} = \text{diag}[\tilde{J}^{(+,+)_{D_{nd}}}, \tilde{J}^{(-,+)_{D_{nd}}}, \tilde{J}^{(1)_{D_{nd}}}, \tilde{J}^{(2)_{D_{nd}}}, \ldots].$$

The transformation matrix $H^{D_{nd}}$ is a permutation matrix that rearranges the Fourier series. To be precise, $\mathbf{w}^{D_{nd}}$ is defined as

$$\mathbf{w}^{D_{nd}} = (\mathbf{u}_{\rho 1}^{T}, \mathbf{u}_{\rho 2}^{T}, \ldots)^{T}, \quad \mathbf{u}_{\rho i} = (u_{\rho_i 1}, u_{\rho_i 2}, \ldots)^{T}$$

by permuting the order of the components of $\mathbf{u}$ by the permutation

$$\begin{pmatrix} 1 & 2 & \cdots \\ \rho_1 & \rho_2 & \cdots \end{pmatrix}, \tag{36}$$

which is defined to be consistent with eqn (35).

When the diagonal blocks are all regular, the system has the trivial D_{nd}-invariant solution. The diagonal block $\tilde{J}^{(+,+)_{D_{nd}}}$ becomes singular at a limit point of the loading parameter f. The block $\tilde{J}^{(-,+)_{D_{nd}}}$ becomes singular at a simple, symmetric bifurcation point, at which a D_n-invariant bifurcation path branches. The block $\tilde{J}^{(j)_{D_{nd}}}$ becomes singular at a simple bifurcation point, which is "symmetric" for $\hat{j}+\hat{n}$ odd and is "asymmetric" for even. Its bifurcation path is $D_{(n/\hat{n})d}$-invariant for $\hat{j}+\hat{n}$ even, and $D_{n/\hat{n}}$-invariant for odd. The categorization of the critical points is given in Table 4.

4.3. *D_n-invariant bifurcation path*

The rule of the bifurcation of the D_n-invariant bifurcation path can be obtained similarly. With respect to the group D_n, the space $\hat{X}$ is decomposed into:

Table 4. Categorization of the critical points of the $D_{\infty h}$-invariant plate
(a) $D_{\infty h}$-invariant main path

μ satisfying $\det \tilde{J}^{\mu}=0$	$(+,+)_{D_{\infty h}}$	$(n,-)_{D_{\infty h}}$
Category of points	Limit point	Bifurcation point
Multiplicity M	1	1
Symmetry of points		Symmetric
Symmetry of solutions	$D_{\infty h}$	D_{nd}

(b) D_{nd}-invariant bifurcation path

μ satisfying $\det \tilde{J}^{\mu}=0$	$(+,+)_{D_{nd}}$	$(-,+)_{D_{nd}}$	$(j)_{D_{nd}}$
Category of points	Limit point	Bifurcation point	Bifurcation point
Multiplicity M	1	1	1
Symmetry of points		Symmetric	Asymmetric, $\hat{j}+\hat{n}$ is even Symmetric, $\hat{j}+\hat{n}$ is odd
Symmetry of solutions	D_{nd}	D_n	$D_{(n/\hat{n})d}$, $\hat{j}+\hat{n}$ is even $D_{n/\hat{n}}$, $\hat{j}+\hat{n}$ is odd

(c) D_n-invariant bifurcation path

μ satisfying $\det \tilde{J}^{\mu}=0$	$(+,+)_{D_n}$	$(-,+)_{D_n}$	$(j)_{D_n}$
Category of points	Limit point	Bifurcation point	Bifurcation point
Multiplicity M	1	1	1
Symmetry of points		Symmetric	Asymmetric, $\hat{n}$ is odd Symmetric, $\hat{n}$ is even
Symmetry of solutions	D_n	$D_{n/2}$	$D_{n/\hat{n}}$

Where $(-,+)_{D_n}$ exists only for n even.

$$\hat{X}=\begin{cases}\hat{X}^{(+,+)_{D_n}}\oplus\left(\bigoplus_{j=1}^{(n-1)/2}\hat{X}^{(j)_{D_n}}\right) & \text{if } n \text{ is odd,}\\ \hat{X}^{(+,+)_{D_n}}\oplus\hat{X}^{(-,+)_{D_n}}\oplus\left(\bigoplus_{j=1}^{n/2-1}\hat{X}^{(j)_{D_n}}\right) & \text{if } n \text{ is even.}\end{cases}\tag{37}$$

In this equation the subspaces of the irreducible representations $(+,-)_{D_n}$ and $(-,-)_{D_n}$ are absent. Each subspace in eqn (37) is spanned by:

$$\hat{X}^{(+,+)_{D_n}}=\operatorname{span}\left[\sin\left(kn\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{k,l=1}^{\infty},$$

$$\hat{X}^{(-,+)_{D_n}}=\operatorname{span}\left[\sin\left(\left(k-\frac{1}{2}\right)n\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{k,l=1}^{\infty},$$

$$\hat{X}^{(j)_{D_n}}=\operatorname{span}\left[\sin\left([(k-1)n+j]\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right),\right.$$

$$\left.\sin\left((kn-j)\frac{\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right)\right]_{k,l=1}^{\infty},\tag{38}$$

and the symmetry of each subspace is labeled by

$$\Sigma(\hat{X}^{(+,+)_{D_n}})=D_n,\quad \Sigma(\hat{X}^{(-,+)_{D_n}})=\Sigma(\hat{X}^{(-,-)_{D_n}})=D_{n/2},\quad \Sigma(\hat{X}^{(j)_{D_n}})=D_{n/\hat{n}}.$$

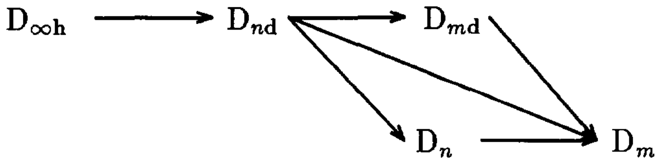

Fig. 5. A lattice of subgroups expressing a bifurcation rule for the $D_{\infty h}$-invariant plate (n/m is odd for $D_{nd} \to D_{md}$).

The block-diagonal form of the tangential stiffness matrix becomes

$$\tilde{J}^{D_n} = (H^{D_n})^T J H^{D_n}$$
$$= \begin{cases} \mathrm{diag}[\tilde{J}^{(+,+)D_n}, \tilde{J}^{(1)D_n}, \ldots, \tilde{J}^{((n-1)/2)D_n}] & \text{if } n \text{ is odd,} \\ \mathrm{diag}[\tilde{J}^{(+,+)D_n}, \tilde{J}^{(-,+)D_n}, \tilde{J}^{(1)D_n}, \ldots, \tilde{J}^{(n/2-1)D_n}] & \text{if } n \text{ is even.} \end{cases}$$

The transformation matrix H^{D_n} is a permutation matrix to be determined in view of eqn (38). When the diagonal blocks are all regular, the system has the trivial D_n-invariant solution. The diagonal block $\tilde{J}^{(+,+)D_n}$ becomes singular at a limit point of the loading parameter f. The block $\tilde{J}^{(-,+)D_n}$ or $\tilde{J}^{(j)D_n}$ becomes singular at a simple bifurcation point, at which a $D_{n/2}$- or $D_{n/\hat{n}}$-invariant bifurcation path branches, respectively. This bifurcation point is "asymmetric" for $\tilde{J}^{(j)D_n}$ with $\hat{n}$ odd and is "symmetric" otherwise.

4.4. *Bifurcation hierarchy*

The assemblage of the rules for the bifurcation of $D_{\infty h}$-, D_{nd}- and D_n-invariant paths presented in the previous subsections leads to the bifurcation rule of the plate shown in Fig. 5. Note that D_{md} is a subgroup of D_{nd} only if n/m is odd. This rule is much simpler than that of a $D_{\infty h}$-equivariant system in Fig. 3, because of the absence of the modes other than $D_{\infty h}$-, D_{nd}- and D_n. Such simplicity is due to the degeneration by the boundaries.

5. BIFURCATION ANALYSIS

A bifurcation analysis was carried out on the simply-supported plate. The numbers of the Fourier series in eqn (22) were chosen to be $m_x = m_y = 6$. The aspect ratio was chosen to be $\alpha = 0.8$ and the depth-thickness ratio to be $\beta \equiv b/t = 200$. For this case, the lattice of subgroups which expresses the bifurcation rule degenerates as shown in Fig. 6, because D_{nd}-invariant bifurcation paths with $n \geqslant 7$ and D_n-invariant ones with $n \geqslant 4$ do to exist. Such degeneration is a kind of discretizing error, which can be reduced to some extent by

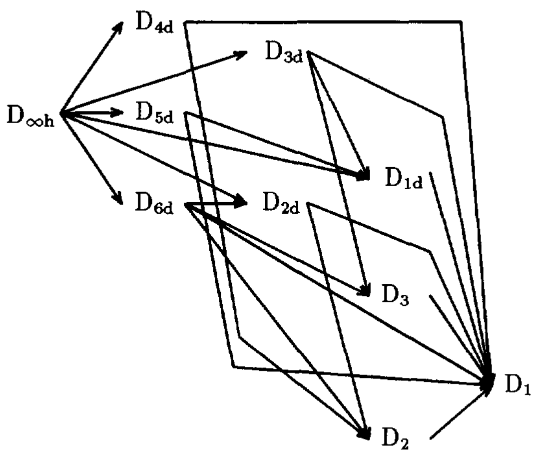

Fig. 6. A lattice of subgroups expressing a bifurcation rule for the plate for $m_x = m_y = 6$.

Table 5. Modes ($m_x = 6$) and symmetries of bifurcation paths

	μ	$(+,+)_{D_{1d}}$	$(-,+)_{D_{1d}}$				
D_{1d}	order	1, 3, 5	2, 4, 6				
	symmetry	D_{1d}	D_1				
	μ	$(+,+)_{D_{2d}}$	$(-,+)_{D_{2d}}$	$(1)_{D_{2d}}$			
D_{2d}	order	2, 6	4	1, 3, 5			
	symmetry	D_{2d}	D_2	D_1			
	μ	$(+,+)_{D_{3d}}$	$(-,+)_{D_{3d}}$	$(1)_{D_{3d}}$	$(2)_{D_{3d}}$		
D_{3d}	order	3	6	1, 5	2, 4		
	symmetry	D_{3d}	D_3	D_{1d}	D_1		
	μ	$(+,+)_{D_{4d}}$	$(1)_{D_{4d}}$	$(2)_{D_{4d}}$	$(3)_{D_{4d}}$		
D_{4d}	order	4	1	2, 6	3, 5		
	symmetry	D_{4d}	D_1	D_2	D_1		
	μ	$(+,+)_{D_{5d}}$	$(1)_{D_{5d}}$	$(2)_{D_{5d}}$	$(3)_{D_{5d}}$	$(4)_{D_{5d}}$	
D_{5d}	order	5	1	2	3	4, 6	
	symmetry	D_{5d}	D_{1d}	D_1	D_{1d}	D_1	
	μ	$(+,+)_{D_{6d}}$	$(1)_{D_{6d}}$	$(2)_{D_{6d}}$	$(3)_{D_{6d}}$	$(4)_{D_{6d}}$	$(5)_{D_{6d}}$
D_{6d}	order	6	1	2	3	4	5
	symmetry	D_{6d}	D_1	D_{2d}	D_3	D_2	D_1
	μ	$(+,+)_{D_1}$					
D_1	order	1, 2, 3, 4, 5, 6					
	symmetry	D_1					
	μ	$(+,+)_{D_2}$	$(-,+)_{D_2}$				
D_2	order	2, 4, 6	1, 3, 5				
	symmetry	D_2	D_1				
	μ	$(+,+)_{D_3}$	$(1)_{D_3}$				
D_3	order	3, 6	1, 2, 4, 5				
	symmetry	D_3	D_1				

choosing the number m_x large and to have many divisors. It is to be noted that the bifurcation rule for this case, which involves a relatively small number of groups, is already this complex. For example, on a D_{1d}-invariant bifurcation path, from eqn (35), one can see that the 1st, 3rd, and 5th modes correspond to the space $X^{(+,+)_{D_{1d}}}$, and the 2nd, 4th, and 6th ones to $X^{(-,+)_{D_{1d}}}$. Table 5 shows the relationship among mode numbers, subspaces, and the symmetry of bifurcation paths obtained in this manner.

Figure 7 shows a result of the bifurcation analysis. When the bifurcated solutions branching toward the positive and negative directions of deflection correspond to the identical physical behavior, only one of them was plotted in this figure for simplicity. The abscissa denotes the normalized deflection at $(x, y) = (0.35a, 0.70b)$, and the ordinate

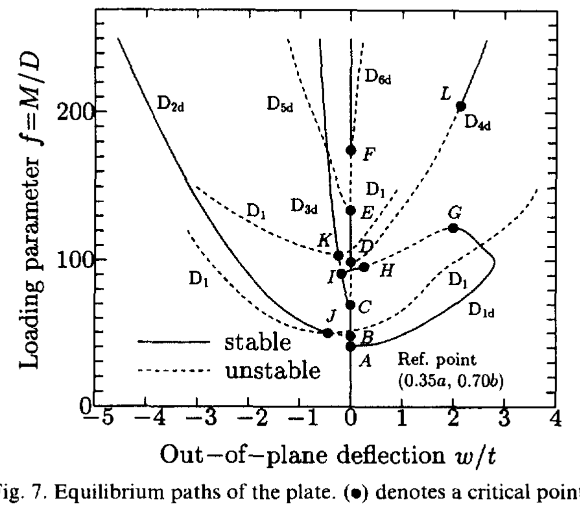

Fig. 7. Equilibrium paths of the plate. (•) denotes a critical point.

indicates the loading parameter f expressing the normalized bending moment. The symmetry groups for the paths are shown in the figure, and critical points are expressed by (●). The solid lines denote stable paths on which all eigenvalues of the tangential stiffness matrix J are positive, and the dashed ones denote unstable ones with one or more negative eigenvalues. The points A, ..., F on the trivial solution $w/t = 0$ are bifurcation points with bifurcation modes of the 1, ..., 6th sine modes in the x-direction, respectively.

We obtained the bifurcation paths from the bifurcation points A, ..., F that are D_{1d}-, ..., D_{6d}-invariant, respectively. By the rule in Table 4(a), these points are simple, symmetric bifurcation points associated, respectively, with the irreducible representations $(n, -)_{D_{\infty h}}$ $(n = 1, \ldots, 6)$.

A D_{1d}-invariant path further branches from the D_{3d}-invariant bifurcation path at the bifurcation point I, and is connected with another D_{1d}-invariant one branching from the main path at the point A. By the rule in Table 4(b), the point I, at which $\hat{j} = 1$ and $\hat{n} = 3$ ($\hat{j} + \hat{n}$ is even) is a simple, asymmetric bifurcation point associated with the irreducible representation $(1)_{D_{nd}}$. D_1-invariant paths branch from the bifurcation point J on the D_{2d}-invariant one and the bifurcation point K on the D_{3d}-invariant one. The point J with $\hat{j} = 1$ and $\hat{n} = 2$ and the point K with $\hat{j} = 2$ and $\hat{n} = 3$ are simple, symmetric bifurcation points because $\hat{j} + \hat{n}$ is odd. No bifurcation takes place on D_1-invariant ones because they cannot lose symmetries any further. The complex bifurcation process presented above does perfectly follow the rule in Fig. 6.

At the course of the bifurcation analysis, the block-diagonalization of the tangential stiffness matrix J was performed. For the $D_{\infty h}$-invariant main path, J is already in a diagonal form as was explained in the previous section. For the D_{nd}-invariant $(n = 1, \ldots, 6)$ bifurcation paths, the subspaces in the isotypic decomposition in eqn (34) are spanned as listed in Table 5. The permutation in eqn (36) was chosen with reference to this table, that is,

$$\begin{cases} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 2 & 6 \end{pmatrix} & \text{for } D_{1d}, D_{2d}, D_{3d}, D_1, D_2, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 2 & 4 & 5 \end{pmatrix} & \text{for } D_3, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 6 & 3 & 5 \end{pmatrix} & \text{for } D_{4d}, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 2 & 3 & 4 & 6 \end{pmatrix} & \text{for } D_{5d}, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} & \text{for } D_{6d}. \end{cases}$$

Figures 8 and 9 show the Jacobian J and $\tilde{J}^{D_{nd}}$, before and after the block-diagonalization, respectively. Here (·) stands for a zero component (∗) for nonzero one. The forms of these matrices are all in agreement with eqn (34). The block-diagonalization method has turned out to be a refined way to understand and categorize the complex bifurcation behavior because the block-diagonal form $\tilde{J}^{D_{nd}}$ of the Jacobian J corresponded to the categorization of critical points. In addition, the method achieves numerical stability during iteration because the block for the main path is not singular even at a bifurcation point.

6. CONCLUSIONS

Although the recursive bifurcation of the rectangular plate turned out to be quite complex, the present method is capable of describing the qualitative aspects of the behavior in a complete manner. The block-diagonalization method is useful in the classification of the critical points and in the stabilization of the iteration. The rule of the bifurcation of a

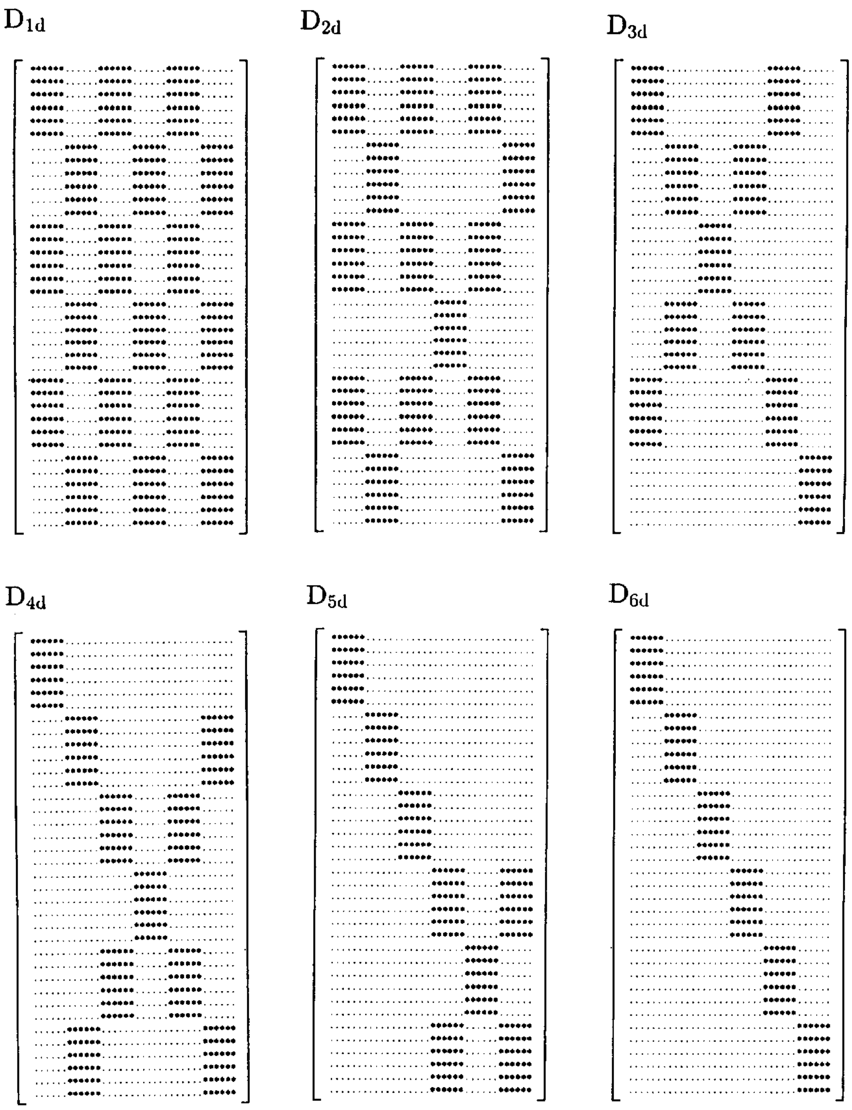

Fig. 8. Distribution of nonzero components of the tangential stiffness matrix before block-diagonalization. (·) stands for a zero component and (*) for nonzero one.

system with "hidden symmetry" undergoes degeneration due to boundary conditions. It is desirable to develop a synthetic theory to describe the mechanism of the degeneration, which is dependent on particular boundary conditions. This paper hopefully serves as a first step for this development.

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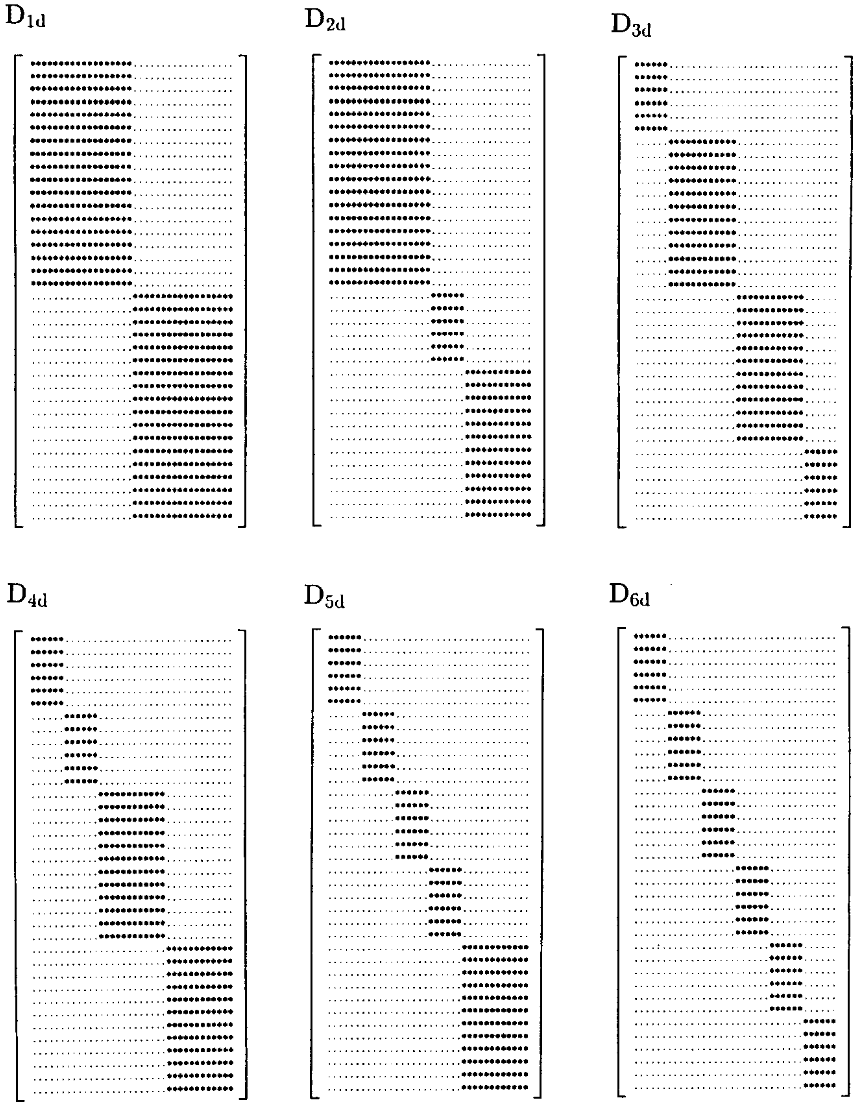

Fig. 9. Distribution of nonzero components of the tangential stiffness matrix $\tilde{J}^{D_{nd}}$ after block-diagonalization. (·) stands for a zero component and (*) for nonzero one.

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APPENDIX

Double bifurcation point of a D_{nd}-equivariant system

The bifurcation behavioral characteristics of the double bifurcation point associated with the jth two-dimensional irreducible representations of the D_{nd}-equivariant system are determined by solving the bifurcation equation (5). The actions of the two-dimensional irreducible representations, labeled by $(1)_{D_{nd}}, (2)_{D_{nd}}, \ldots, (n-1)_{D_{nd}}$ are defined by†

$$T^{(j)_{D_{nd}}}(\sigma_h c(\pi/n)) \equiv Q = -\begin{bmatrix} \cos \pi \frac{j}{n} & -\sin \pi \frac{j}{n} \\ \sin \pi \frac{j}{n} & \cos \pi \frac{j}{n} \end{bmatrix},$$

$$T^{(j)_{D_{nd}}}(\sigma_h \sigma_v) \equiv P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = 1, \ldots, n-1. \tag{A1}$$

From eqn (A1) we note that

$$Q^{2\hat{n}} = I_2,$$

$$Q^{\hat{n}} = (-1)^{\hat{n}} \begin{bmatrix} \cos \pi \frac{\hat{j}}{\hat{n}} & -\sin \pi \frac{\hat{j}}{\hat{n}} \\ \sin \pi \frac{\hat{j}}{\hat{n}} & \cos \pi \frac{\hat{j}}{\hat{n}} \end{bmatrix}^{\hat{n}} = (-1)^{\hat{j}+\hat{n}} = \begin{cases} -I_2 & \text{if } \hat{j}+\hat{n} \text{ is odd,} \\ I_2 & \text{if } \hat{j}+\hat{n} \text{ is even,} \end{cases}$$

where I_2 is the 2×2 identity matrix and $\hat{j} = j/\gcd(j,n)$ and $\hat{n} = n/\gcd(j,n)$. Here $\gcd(j,n)$ denotes the greatest common divisor of j and n, and $\hat{j}$ and $\hat{n}$ are relatively prime. The symmetry of the kernel space, accordingly, is labeled by

$$G^{(j)} = \begin{cases} \langle (c(\pi/n))^{2\hat{n}} \rangle = \langle c(\pi 2\hat{n}/n) \rangle = C_{n/\hat{n}} & \text{if } \hat{j}+\hat{n} \text{ is odd,} \\ \langle (\sigma_h c(\pi/n))^{\hat{n}} \rangle = \langle \sigma_h c(\pi \hat{n}/n) \rangle = S_{2n/\hat{n}} & \text{if } \hat{j}+\hat{n} \text{ is even,} \end{cases} \quad j = 1, \ldots, n-1. \tag{A2}$$

Let $\mathbf{e}_1^c$ and $\mathbf{e}_2^c$ be two orthonormal real eigenvectors in the kernel space at the double point. Then an arbitrary eigenvector $(\mathbf{e}^c)^*$ in this subspace is expressed as

† The irreducible representation matrix $T^{(j)_{D_{nd}}}(\sigma_h \sigma_v)$, which is not unique, is elaborately chosen to be P, instead of $-P$, so as to simplify the resulting formulas.

$$(\mathbf{e}^c)^* = 2\tilde{w}_1\mathbf{e}_1^c + 2\tilde{w}_2\mathbf{e}_2^c.$$

With the use of a complex variable $z = \tilde{w}_1 + i\tilde{w}_2$, the two-dimensional kernel space can be identified with the space of variables z and $\bar{z}$, where i denotes the imaginary unit and $\bar{(\cdot)}$ denotes the complex conjugate of the relevant variable. It is emphasized that not all of these eigenvectors, in reality, specify the directions of bifurcation paths, but $(z, \bar{z})$ should satisfy two-dimensional bifurcation equations ($M = 2$ in eqn (5)),

$$\tilde{F}_i(z, \bar{z}) = 0, \quad i = 1, 2. \tag{A3}$$

If we put

$$\tilde{F}(\tilde{f}, z, \bar{z}) = \tilde{F}_1(\tilde{f}, \tilde{w}_1, \tilde{w}_2) + i\tilde{F}_2(\tilde{f}, \tilde{w}_1, \tilde{w}_2),$$

where $\tilde{f}$ is an increment of f from its value at the double point ($\tilde{f} = 0$ at the point), we see that eqn (A3) is equivalent to a complex equation

$$\tilde{F}(\tilde{f}, z, \bar{z}) = 0, \tag{A4}$$

since $\tilde{F}_1$ and $\tilde{F}_2$ are real. Suppose we can expand $\tilde{F}$ as

$$\tilde{F}(\tilde{f}, z, \bar{z}) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{pq}(\tilde{f}) z^p \bar{z}^q. \tag{A5}$$

Since $(\tilde{f}, \tilde{w}_1, \tilde{w}_2) = (0, 0, 0)$ corresponds to the double critical point, we have

$$A_{00}(0) = A_{10}(0) = A_{01}(0) = 0. \tag{A6}$$

The D_{nd}-equivariance (6) at a group-theoretic point is expressed as follows. Let D_{nd} act on $(z, \bar{z})$ via

$$\begin{aligned} \sigma_h\sigma_v \cdot z = \bar{z}, \quad \sigma_h\sigma_v \cdot \bar{z} = z, \\ \sigma_h c(\pi/n) z = -\omega z, \quad \sigma_h c(\pi/n)\bar{z} = -\omega \bar{z}, \end{aligned} \tag{A7}$$

where

$$\omega = \exp i\pi\frac{j}{n} = \exp i\pi\frac{\hat{j}}{\hat{n}}.$$

Then the D_{nd}-equivariance (6) is equivalent to

$$\overline{\tilde{F}(\tilde{f}, z, \bar{z})} = \tilde{F}(\tilde{f}, \bar{z}, z). \tag{A8}$$

$$-\omega\tilde{F}(\tilde{f}, z, \bar{z}) = \tilde{F}(\tilde{f}, -\omega z, -\bar{\omega}\bar{z}). \tag{A9}$$

Substitution of eqn (A5) into eqn (A8) yields

$$A_{pq}(\tilde{f}) \text{ is real}, \quad p, q = 0, 1, \ldots. \tag{A10}$$

From eqn (A9) we see that

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{pq}(\tilde{f}) z^p \bar{z}^q \left\{(-1)^{p-q-1} \exp\left[i\pi\frac{(p-q-1)\hat{j}}{\hat{n}}\right] - 1\right\} = 0.$$

Hence,

$$\begin{aligned} A_{pq}(\tilde{f}) = 0 \quad \text{unless} \quad p-q-1 = k\hat{n} \quad \text{and} \quad p-q-1+k\hat{j} = \text{even}, \\ p, q = 0, 1, \ldots; \quad k = 0, \pm 1, \pm 2, \ldots, \end{aligned}$$

that is,

$$A_{pq} \text{ is nonzero for } p-q-1 = \begin{cases} 2k\hat{n} & \text{if } \hat{j}+\hat{n} \text{ is odd}, \\ k\hat{n} & \text{if } \hat{j}+\hat{n} \text{ is even}. \end{cases} \tag{A11}$$

Equations (A10) and (A11) are the conditions for the D_{nd}-symmetry. The properties of the bifurcated solutions satisfying the complex bifurcation equation (A4) with these conditions, accordingly, depends on the parity of $\hat{j}+\hat{n}$.

A.1. $\hat{j}+\hat{n}$ *is even*

For $\hat{j}+\hat{n}$ is even ($\hat{n} \geqslant 3$), under the condition (A11), eqn (A5) is rewritten as

$$\tilde{F}(\tilde{f}, z, \bar{z}) = \sum_{q=0}^{\infty} A_{q+1,q}(\tilde{f}) z^{q+1}\bar{z}^{q} + \sum_{k=1}^{\infty}\sum_{q=0}^{\infty} [A_{q+1+k\hat{n},q}(\tilde{f}) z^{q+1+k\hat{n}}\bar{z}^{q} + A_{q,q-1+k\hat{n}}(\tilde{f}) z^{q}\bar{z}^{q-1+k\hat{n}}]. \quad \text{(A12)}$$

The equilibrium eqn (A4) has the trivial solution $z = 0$, corresponding to the D_{nd}-symmetric main path. Its non-trivial solution is determined from $\tilde{F}/z = 0$. If we put

$$\hat{F}(\tilde{f}, r, \theta) = \frac{\tilde{F}(\tilde{f}, r\exp(\mathrm{i}\theta), r\exp(-\mathrm{i}\theta))}{r\exp(\mathrm{i}\theta)}$$

using the polar coordinates $z = r\exp(\mathrm{i}\theta)$ and eqn (A12), we have

$$\mathscr{R}(\hat{F}) = \sum_{q=0}^{\infty} A_{q+1,q}(\tilde{f}) r^{2q}$$
$$+ \sum_{k=1}^{\infty}\sum_{q=0}^{\infty} [A_{q+1+k\hat{n},q}(\tilde{f}) r^{2q+k\hat{n}} + A_{q,q-1+k\hat{n}}(\tilde{f}) r^{2(q-1)+k\hat{n}}] \cos(k\hat{n}\theta),$$

$$\mathscr{I}(\hat{F}) = \sum_{k=1}^{\infty}\sum_{q=0}^{\infty} [A_{q+1+k\hat{n},q}(\tilde{f}) r^{2q+k\hat{n}} - A_{q,q-1+k\hat{n}}(\tilde{f}) r^{2(q-1)+k\hat{n}}] \sin(k\hat{n}\theta). \quad \text{(A13)}$$

[A_{pq} is real by eqn (A10)]. The non-trivial solution is to satisfy equation $\mathscr{I}(\hat{F}) = 0$ and hence $\sin(\hat{n}\theta) = 0$. Therefore,

$$\theta = \alpha_k = \frac{\pi(k-1)}{\hat{n}}, \quad k = 1, \ldots, 2\hat{n}$$

is necessary for the existence of non-trivial solutions.

If we put

$$\hat{F}_i(\tilde{f}, r) = \hat{F}(\tilde{f}, r, \alpha_i), \quad i = 1, 2,$$

we see from eqn (A13) that

$$\hat{F}_i(\tilde{f}, r) = \hat{F}(\tilde{f}, r, \alpha_{2(k-1)+i}), \quad k = 1, \ldots, \hat{n}, \quad i = 1, 2. \quad \text{(A14)}$$

It can be proven by the implicit function theorem that, for each i, $\hat{F}_i(\tilde{f}, r) = 0$ can be solved generically for $\tilde{f}$ as $\tilde{f} = f_i(r)$ in the neighborhood of the double point $(\tilde{f}, r) = (0, 0)$. Therefore, the number of bifurcation paths equals $2\hat{n}$.

The above argument [see eqn (A14)] shows the existence of two distinct sets of bifurcation paths denoted by $f_1(r)$ and $f_2(r)$. Hence, the $2\hat{n}$ bifurcation paths are divided into two physically independent paths. Every other bifurcation path in θ-direction is associated with a physically independent path with the same $\tilde{f}$ vs r curve. A pair of paths which branch in opposite directions $\theta = \alpha_k$ and $\theta = \alpha_k + \pi = \alpha_{k+\hat{n}}$ ($k = 1, \ldots, 2\hat{n}$) correspond to different functions f_i ($i = 1, 2$); the bifurcation point in this sense is defined to be asymmetric.

The symmetry of the system on the bifurcation paths is determined. The paths branching toward α_k ($k = 1, \ldots, 2\hat{n}$) are associated with

$$z = r\exp(\mathrm{i}\pi\alpha_k) = r\exp\left(\mathrm{i}\pi\frac{k-1}{\hat{n}}\right), \quad k = 1, \ldots, 2\hat{n}. \quad \text{(A15)}$$

Let us show the existence of an integer p satisfying

$$\sigma_{\mathrm{h}}\sigma_{\mathrm{v}}(\sigma_{\mathrm{h}}c(\pi/n))^{2p} \cdot z = z \quad \text{(A16)}$$

which implies that z is invariant with respect to the action of

$$\sigma_{\mathrm{h}}\sigma_{\mathrm{v}}(\sigma_{\mathrm{h}}c(\pi/n))^{2p} = \sigma_{\mathrm{h}}\sigma_{\mathrm{v}}c(\pi 2p/n).$$

The substitution of eqns (A7) and (A15) into eqn (A16) leads to

$$\exp\left[2\mathrm{i}\pi\frac{p\hat{j} + (k-1)\hat{n}}{\hat{n}}\right] = 1,$$

that is

$$p\hat{j} + (k-1)\hat{n} = m\hat{n}. \quad \text{(A17)}$$

As is well-known, integers p and m satisfying this equation exist because $\hat{j}$ and $\hat{n}$ are relatively prime. The symmetry of this path, therefore, is generated by the element in eqn (A2) and the element $\sigma_h\sigma_v c(\pi 2p/n)$ $(k = 1, \ldots, 2\hat{n})$ for p satisfying eqn (A17) and, hence, is labeled by

$$\langle \sigma_h\sigma_v c(\pi 2p/n), \sigma_h c(\pi\hat{n}/n)\rangle = D_{(n/\hat{n})d}.$$

Here, it should be noted that $\sigma_v c(\pi 2p/n)$ was replaced with σ_v because the location of the vertical reflection plane is arbitrary in the context of the Schoenflies notation.

A.2. $\hat{j}+\hat{n}$ *is odd*

For $\hat{j}+\hat{n}$ is odd $(\hat{n} \geq 2)$, under the condition (A11), $k\hat{n}$ in eqns (A12) and (A13) is to be replaced with $2k\hat{n}$. The non-trivial solution, accordingly, is to satisfy $\sin(2\hat{n}\theta) = 0$. Therefore, $4\hat{n}$ bifurcation paths exist towards the directions of

$$\theta = \alpha_k = \frac{\pi(k-1)}{2\hat{n}}, \quad k = 1, \ldots, 4\hat{n}.$$

Similar to the case of $\hat{j}+\hat{n}$ even, the $4\hat{n}$ bifurcation paths are divided into two physically independent paths and every other bifurcation path in θ-direction is associated with an independent path. A pair of paths branching in opposite directions $\theta = \alpha_k$ and $\theta = \alpha_k + \pi = \alpha_{k+2\hat{n}}$ $(k = 1, \ldots, 2\hat{n})$ represent the same f vs r curve; the bifurcation point in this sense is defined to be symmetric.

The bifurcation path branching in the direction of α_k $(k = 1, \ldots, 4\hat{n})$ is associated with $z = r\exp[i\pi(k-1)/(2\hat{n})]$. The symmetry of the system on the bifurcation paths is determined. Let us examine the existence of an integer p satisfying

$$\sigma_h\sigma_v(\sigma_h c(\pi/n))^p \cdot z = z, \tag{A18}$$

which implies that z is invariant with respect to the action of

$$\sigma_h\sigma_v(\sigma_h c(\pi/n))^p = \begin{cases} \sigma_v c(\pi p/n) & \text{if } p \text{ is odd,} \\ \sigma_h\sigma_v c(\pi p/n) & \text{if } p \text{ is even.} \end{cases} \tag{A19}$$

The substitution of eqn (A7) and $z = r\exp[i\pi(k-1)/(2\hat{n})]$ into eqn (A18) gives

$$\exp\left\{i\pi\left[p\left(1+\frac{\hat{j}}{\hat{n}}\right)+\frac{k-1}{\hat{n}}\right]\right\} = 1,$$

that is

$$p(\hat{j}+\hat{n})+k-1 = 2m\hat{n}.$$

It is easy to see that the integer p satisfying this equation exists and has the same parity as $k-1$. The symmetry of the bifurcation path branching in the direction of α_k $(k = 1, \ldots, 4\hat{n})$ is generated by and the element $c(\pi 2\hat{n}/n)$ in eqn (A2) and the element $\sigma_h\sigma_v c(\pi p/n)$ in eqn (A18). Its symmetry group, accordingly, is

$$\langle c(\pi 2\hat{n}/n), \sigma_h\sigma_v c(\pi p/n)\rangle = \begin{cases} C_{(n/\hat{n})v} & \text{if } k-1 \text{ is odd,} \\ D_{n/\hat{n}} & \text{if } k-1 \text{ is even} \end{cases}$$

by eqn (A19). The symmetry of the bifurcation paths thus alternates in the θ-direction between $C_{(n/\hat{n})v}$ and $D_{n/\hat{n}}$.